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ON EFFICIENCY OF FIRST AND SECOND ORDER

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On efficiency of first and second order^{*)}

, by

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ABSTRACT

It has been noted by a number of authors that if two tests are asymptotically efficient for the same testing problem, then typically their powers will not only agree to first but also to second order. A general result of this type was given by Pfanzagl (1979) in a paper entitled "First order efficiency implies second order efficiency". Because of their technical nature, however, these contributions give little insight into the nature of this phenomenon. The purpose of the present paper is to provide an intuitive understanding of the phenomenon by proving a simple theorem of this kind under mild assumptions.

KEY WORDS & PHRASES: *Asymptotic efficiency of tests, second order efficiency, deficiency*

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1. INTRODUCTION

For $N = 1, 2, \dots$, we consider an experiment with outcome X_N taking values in an arbitrary sample space. Let $P_{N,0}$ and $P_{N,1}$ be two possible distributions of X_N , with densities $p_{N,0}$ and $p_{N,1}$ with respect to some dominating measure μ_N . We shall write $E_{N,0}$ and $E_{N,1}$ for expectations under $P_{N,0}$ and $P_{N,1}$ respectively. Define the logarithm of the likelihood ratio by

$$\Lambda_N = \log \frac{p_{N,1}(X_N)}{p_{N,0}(X_N)}$$

with the usual conventions for vanishing $p_{N,0}$ and / or $p_{N,1}$.

Now consider a sequence $\alpha_N \in (0, 1)$ and let $\phi_N(\Lambda_N, \alpha_N)$ denote the test function of the most powerful level- α_N test for $P_{N,0}$ against $P_{N,1}$, thus

$$\phi_N(\Lambda_N, \alpha_N) = \begin{cases} 0 & \text{for } \Lambda_N < c_N(\alpha_N) \\ 1 & \text{for } \Lambda_N > c_N(\alpha_N) \end{cases},$$

with

$$E_{N,0} \phi_N(\Lambda_N, \alpha_N) = \alpha_N$$

$$E_{N,1} \phi_N(\Lambda_N, \alpha_N) = \pi_N^*(\alpha_N),$$

where $\pi_N^*(\alpha_N)$ is the maximum attainable power against $P_{N,1}$ at level α_N .

For $N = 1, 2, \dots$, let Z_N be a random variable depending only on the outcome X_N of the N -th experiment and let $\psi_N(Z_N, \alpha_N)$ denote the test function of the level- α_N right-sided test based on the statistic Z_N , i.e.

$$\psi_N(Z_N, \alpha_N) = \begin{cases} 0 & \text{for } Z_N < d_N(\alpha_N) \\ 1 & \text{for } Z_N > d_N(\alpha_N) \end{cases}.$$

We have

$$E_{N,0} \psi_N(Z_N, \alpha_N) = \alpha_N$$

$$E_{N,1} \psi_N(Z_N, \alpha_N) = \pi_N(\alpha_N),$$

where $\pi_N(\alpha_N)$ is the power of this test against $P_{N,1}$.

For a sequence $\tau_N \in (0,1]$, we shall say that the sequence of level- α_N tests $\psi_N(Z_N, \alpha_N)$ is τ_N -efficient if, for $N \rightarrow \infty$,

$$\pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(\tau_N) .$$

In a more usual terminology first and second order efficiency correspond to τ_N -efficiency with $\tau_N = 1$ and $\tau_N = N^{-\frac{1}{2}}$ respectively.

Finally, let us define for $N = 1, 2, \dots$,

$$\Delta_N = \begin{cases} 0 & \text{if } Z_N = \Lambda_N = \pm \infty \\ Z_N - \Lambda_N & \text{otherwise ,} \end{cases}$$

and let us denote the indicator function of a set B by 1_B .

Having established our notation we shall now give an informal description of the phenomenon we wish to study. Let us think of N as denoting sample size, i.e. N is the number of independent random variables involved in the N -th testing problem. We are interested primarily in sequences of testing problems where $\alpha_N \geq \varepsilon$ and $\pi_N^*(\alpha_N) \leq 1 - \varepsilon$ for some $\varepsilon > 0$ and all N . Such sequences exist if it is impossible to discriminate perfectly between $P_{N,0}$ and $P_{N,1}$ even as $N \rightarrow \infty$ and this is true if $P_{N,0}$ and $P_{N,1}$ are contiguous. A sufficient condition for contiguity - and one which is often fulfilled in this case - is asymptotic normality of Λ_N both under $P_{N,0}$ and $P_{N,1}$. But if Λ_N is asymptotically normal, it will usually also be possible to obtain an Edgeworth expansion for its distribution function under $P_{N,0}$ and $P_{N,1}$ and this will yield a similar expansion for the power of the test based on Λ_N , viz.

$$(1.1) \quad \pi_N^*(\alpha_N) = c_0 + c_1 N^{-\frac{1}{2}} + o(N^{-\frac{1}{2}}) .$$

Typically the remainder term on the right in (1.1) will be $O(N^{-1})$.

Suppose that the sequence of tests $\psi_N(Z_N, \alpha_N)$ is asymptotically efficient to first order, or 1-efficient in our terminology. For most statistical problems such 1-efficient tests abound. They are usually based on statistics Z_N that closely resemble Λ_N . Typically $\Delta_N = Z_N - \Lambda_N$ tends to zero in probability both under $P_{N,0}$ and $P_{N,1}$ and in the situation we have described so far, this suffices to ensure 1-efficiency. Of course, these 1-efficient tests can also be based on statistics Z_N which don't resemble Λ_N at all, because the test statistic associated with a test is by no means unique. However, we shall not be concerned with such alternative representations and suppose that $\Delta_N \rightarrow 0$ in

$P_{N,0}$ - and $P_{N,1}$ - probability.

Recalling that N is the sample size, we note that one often finds that a sequence of random variables Δ_N tending to zero in probability does so at the rate of $N^{-\frac{1}{2}}$. Thus, for 1-efficient tests, $N^{\frac{1}{2}} \Delta_N$ will typically be bounded in probability both under $P_{N,0}$ and $P_{N,1}$. Hence, one may expect to be able to establish Edgeworth expansions for the distribution functions of Z_N under $P_{N,0}$ and $P_{N,1}$, which differ from those for Δ_N only in the term of order $N^{-\frac{1}{2}}$ and in those of smaller order. This yields a similar expansion for the power of the test based on Z_N ,

$$(1.2) \quad \pi_N(\alpha_N) = c_0 + c'_1 N^{-\frac{1}{2}} + o(N^{-\frac{1}{2}}),$$

where the remainder term on the right will typically be $O(N^{-1})$ or of slightly larger order. The fact that the leading terms in expansions (1.1) and (1.2) have the same value c_0 reflects the 1-efficiency of the sequence $\psi_N(Z_N, \alpha_N)$. There would seem to be no reason a priori to expect that also $c_1 = c'_1$, which would entail $N^{-\frac{1}{2}}$ -efficiency or efficiency of second order.

However, in those cases where expansions (1.1) and (1.2) were explicitly computed, one does indeed find that $c_1 = c'_1$ and hence that the sequence $\psi_N(Z_N, \alpha_N)$ is $N^{-\frac{1}{2}}$ -efficient. This phenomenon was noticed in Pfanzagl (1973), (1975) and Chibisov (1974) for a number of tests for the parametric one-sample problem and in Bickel and Van Zwet (1978) for rank tests for the nonparametric two-sample problem. Some tests for the one-sample problem for the case where nuisance parameters are present were considered in Chibisov (1973) and Pfanzagl (1974) and also found to be $N^{-\frac{1}{2}}$ -efficient. Finally, it was shown in Pfanzagl (1979) that first order efficiency forces second order efficiency for a large class of one-sample tests in the presence of nuisance parameters. With an appropriate definition of efficiency a similar result was obtained for estimators.

In each of these contributions, $N^{-\frac{1}{2}}$ -efficiency is established by imposing the conditions needed to obtain expansions (1.1) and (1.2) and then checking that these expansions are in fact identical. This method of proof coupled with its extreme technicality makes an intuitive understanding of the phenomenon rather difficult. The purpose of the present paper is to provide such an intuitive understanding by proving a simple theorem of this kind under rather mild assumptions. Since our aim is to provide insight rather than generality, we shall only be concerned with the simple hypothesis testing problem described

above and avoid the technicalities inherent in the treatment of nuisance parameters and estimation problems, although extension to these situations is certainly possible.

In section 2 we present our result, discuss its meaning and explain why it is true. A formal proof of the theorem is given in section 3. Though this proof is straightforward, the non-mathematically inclined may wish to skip it.

2. DISCUSSION OF THE RESULT

In the notation introduced in the previous section we have the following result.

THEOREM

Suppose that

$$(2.1) \quad \liminf_N \alpha_N > 0 ,$$

and that there exists $A > 0$ such that for every $x_0 \in \mathbb{R}$, every $\gamma > 0$ and $N \rightarrow \infty$,

$$(2.2) \quad \sup_{x \leq x_0} P_{N,0}(x - \tau_N^{\frac{1}{2}} \leq \Lambda_N \leq x) = o(\tau_N^{\frac{1}{2}}) ,$$

$$(2.3) \quad E_{N,0} |\Delta_N| 1_{(\gamma \tau_N^{\frac{1}{2}}, A)}(|\Delta_N|) = o(\tau_N) ,$$

$$(2.4) \quad P_{N,0}(\Delta_N \geq A) = o(\tau_N) ,$$

$$(2.5) \quad P_{N,1}(\Delta_N \leq -A) = o(\tau_N) .$$

Then the sequence of tests $\psi_N(Z_N, \alpha_N)$ is τ_N -efficient.

Let us briefly discuss the conditions of this theorem. First of all, assumption (2.2) is clearly satisfied for any sequence τ_N if the distributions of Λ_N under $P_{N,0}$ possess uniformly bounded densities. More generally, (2.2) will hold if the distribution functions F_N of Λ_N under $P_{N,0}$ can be approximated with a uniform error of order $\tau_N^{\frac{1}{2}}$ by distribution functions G_N with uniformly bounded densities, i.e. if $\sup_x |F_N(x) - G_N(x)| = o(\tau_N^{\frac{1}{2}})$.

This is certainly the case when F_N has a normal approximation or an Edgeworth expansion with the required error.

If $\tau_N \rightarrow 0$, assumption (2.2) clearly implies that the distributions of Λ_N under $P_{N,0}$ don't tend to a degenerate limit. In view of this, conditions (2.3) - (2.4) serve to ensure that under $P_{N,0}$, $|\Delta_N|$ is small compared to the variation of Λ_N . Note that these conditions refer only to values of $|\Delta_N|$ which exceed $\gamma\tau_N^{\frac{1}{2}}$ and that they are satisfied if the distribution of Δ_N under $P_{N,0}$ either assigns probability 1 to a set where $|\Delta_N| = o(\tau_N^{\frac{1}{2}})$, or has at most very small tails outside that range. Thus, roughly speaking, Δ_N is required to be $o(\tau_N^{\frac{1}{2}})$ under $P_{N,0}$; under $P_{N,1}$ condition (2.5) is even weaker, but this is to a certain extent artificial and is due to our efforts to replace conditions under $P_{N,1}$ as much as possible by conditions under $P_{N,0}$ which will usually be easier to verify. It follows that one can't hope to say more about the differences between the distribution functions of Λ_N and Z_N under $P_{N,0}$ and $P_{N,1}$ than that they are $o(\tau_N^{\frac{1}{2}})$. One would therefore expect to be able to prove that $\pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(\tau_N^{\frac{1}{2}})$ but, somewhat surprisingly, the conclusion of the theorem is the stronger statement that $\pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(\tau_N)$. The condition that roughly Δ_N is $o(\tau_N^{\frac{1}{2}})$ under $P_{N,0}$ cannot essentially be improved. By taking $\Delta_N = \Delta\tau_N^{\frac{1}{2}}$ where Δ is independent of Λ_N for every N , one easily constructs examples where $\pi^*(\alpha_N) - \pi(\alpha_N)$ is of exact order τ_N .

A formal proof of the theorem will be given in section 3. At this point we shall be content to provide an intuitive explanation of the result by sketching the proof for the special case where there exist numbers δ_N such that for $N = 1, 2, \dots$,

$$(2.6) \quad |\Delta_N| < \delta_N = o(\tau_N^{\frac{1}{2}}).$$

By making this assumption we avoid certain technicalities and bring out the essential simplicity of the proof.

Let us write c_N and d_N for $c_N(\alpha_N)$ and $d_N(\alpha_N)$ respectively. Since the tests $\phi_N(\Lambda_N, \alpha_N)$ and $\psi_N(Z_N, \alpha_N)$ have the same level α_N , (2.6) clearly implies that we may assume that $|c_N - d_N| < \delta_N$. Invoking (2.6) once more, we see that if $\Lambda_N \geq c_N$ and $Z_N \leq d_N$, then $d_N - \delta_N \leq \Lambda_N \leq d_N + \delta_N$; the same conclusion holds if $\Lambda_N \leq c_N$ and $Z_N \geq d_N$. It follows that on the set where $\phi_N(\Lambda_N, \alpha_N) \neq \psi_N(Z_N, \alpha_N)$ we have $|\Lambda_N - d_N| \leq \delta_N$, and again because both tests have level α_N we find with the aid of (2.2),

$$\begin{aligned}
\pi_N^*(\alpha_N) - \pi_N(\alpha_N) &= E_{N,1}(\phi_N(\Lambda_N, \alpha_N) - \psi_N(Z_N, \alpha_N)) \\
&= E_{N,0}(e^{\Lambda_N - d_N})(\phi_N(\Lambda_N, \alpha_N) - \psi_N(Z_N, \alpha_N)) \\
&\leq e^{d_N} (e^{\delta_N - 1}) P_{N,0}(|\Lambda_N - d_N| \leq \delta_N) \\
&= O(\delta_N \tau_N^{\frac{1}{2}}) = o(\tau_N) .
\end{aligned}$$

Note that we need to have d_N bounded above, but this is an easy consequence of (2.1) and (2.6). The above sketch should make it clear that the essential thing which makes the theorem work is that not only do $\phi_N(\Lambda_N, \alpha_N)$ and $\psi_N(Z_N, \alpha_N)$ resemble each other closely, but that also Λ_N is almost constant on the set where they differ.

Let us finally discuss the relevance of the theorem to the problem of first and second order efficiency. As was pointed out in section 1, first order efficiency of $\psi_N(Z_N, \alpha_N)$ will typically imply that $N^{\frac{1}{2}}\Delta_N$ is bounded in probability both under $P_{N,0}$ and $P_{N,1}$. Also, $E_{N,0}|N^{\frac{1}{2}}\Delta_N|^r$ and $E_{N,1}|N^{\frac{1}{2}}\Delta_N|^r$ will usually be bounded for some $r > 1$ and this (or even uniform integrability of $|N^{\frac{1}{2}}\Delta_N|$ under $P_{N,0}$ and $P_{N,1}$) is amply sufficient to ensure that assumptions (2.3) - (2.5) are satisfied for $\tau_N = N^{-\frac{1}{2}}$. But then the theorem ensures that $\psi_N(Z_N, \alpha_N)$ is efficient to second order under the very mild conditions (2.1) and (2.2) for $\tau_N = N^{-\frac{1}{2}}$.

An examination of our proof shows that if we replace (2.6) by

$$(2.7) \quad |\Delta_N| \leq M\tau_N^{\frac{1}{2}}$$

for $M < \infty$ and $N = 1, 2, \dots$, then we obtain the conclusion

$$(2.8) \quad \pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(\tau_N) .$$

Taking $\tau_N = N^{-1}$ we conclude that if $|N^{\frac{1}{2}}\Delta_N|$ is bounded and if e.g. the distribution of Λ_N under $P_{N,0}$ tends to normality at the rate of $N^{-\frac{1}{2}}$, then $\pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(N^{-1})$. This means that the tests based on Z_N have a finite deficiency in the sense of Hodges and Lehmann (1970). That is, if we let $\alpha_N = \alpha \in (0, 1)$ for $N = 1, 2, \dots$ and define N' to be the smallest integer for which $\pi_{N'}(\alpha) \geq \pi_N^*(\alpha)$ then $\limsup_N (N' - N) < \infty$.

Assumption (2.7) is of course not necessary to obtain (2.8) and $N^{-\frac{1}{2}}$ -efficiency is frequently coupled with finite deficiency. However, the coupling is not inevitable. An example - in a nuisance parameter context - is provided by the normal scores test studied in Bickel and Van Zwet (1978). Note that this bears out what we have said about the order of the remainder terms in (1.1) and (1.2).

3. PROOF OF THE THEOREM

Take a sequence $\alpha_N \in (0,1)$ satisfying (2.1) and write $c_N = c_N(\alpha_N)$ and $d_N = d_N(\alpha_N)$. If $d_N = -\infty$ for some N then

$$\begin{aligned} 1 - \pi_N(\alpha_N) &\leq P_{N,1}(Z_N = -\infty) \leq P_{N,1}(\Lambda_N = -\infty) + P_{N,1}(\Delta_N = -\infty) \\ &= P_{N,1}(\Delta_N = -\infty) = o(\tau_N) \end{aligned}$$

because of (2.5), and $\psi_N(Z_N, \alpha_N)$ is clearly τ_N -efficient. Take A as in (2.3) - (2.5). Then

$$\begin{aligned} \alpha_N &\leq P_{N,0}(Z_N \geq d_N) \leq P_{N,0}(\Lambda_N \geq d_N - A) + P_{N,0}(\Delta_N \geq A) \\ &\leq e^{-d_N + A} P_{N,1}(\Lambda_N \geq d_N - A) + o(\tau_N) \leq e^{-d_N + A} + o(\tau_N) \end{aligned}$$

because of (2.4). In view of (2.1) it is therefore no loss of generality to assume that for some $D < \infty$ and all N ,

$$(3.1) \quad -\infty < d_N \leq D.$$

Define

$$\bar{\Delta}_N = \begin{cases} d_N - \Lambda_N & \text{if } \Lambda_N < d_N \leq Z_N \text{ or } Z_N \leq d_N < \Lambda_N \\ 0 & \text{otherwise.} \end{cases}$$

Obviously,

$$(3.2) \quad \bar{\Delta}_N^+ \leq \Delta_N^+, \quad \bar{\Delta}_N^- \leq \Delta_N^-,$$

where $x^+ = x \vee 0$ and $x^- = (-x) \vee 0$ denote the positive and negative parts of a number x .

Let α'_N be such that $d_N = c_N(\alpha'_N)$. Then

$$\begin{aligned}
 & \{\pi_N^*(\alpha_N) - \pi_N(\alpha_N)\} + \{\pi_N^*(\alpha'_N) - \pi_N^*(\alpha_N) - e^{d_N}(\alpha'_N - \alpha_N)\} \\
 &= \pi_N^*(\alpha'_N) - \pi_N(\alpha_N) - e^{d_N}(\alpha'_N - \alpha_N) \\
 &= E_{N,0} \{\phi_N(\Lambda_N, \alpha'_N) - \psi_N(Z_N, \alpha_N)\} (e^{\Lambda_N} - e^{d_N}) \\
 &= E_{N,0} |\phi_N(\Lambda_N, \alpha'_N) - \psi_N(Z_N, \alpha_N)| |e^{\Lambda_N} - e^{d_N}| \\
 &\leq e^{d_N} E_{N,0} |e^{-\bar{\Delta}_N} - 1|
 \end{aligned}$$

since $\{\phi_N(\Lambda_N, \alpha'_N) - \psi_N(Z_N, \alpha_N)\}$ is nonnegative (nonpositive) if $(\Lambda_N - d_N)$ is positive (negative) and equals zero if $\bar{\Delta}_N = 0$ and $(\Lambda_N - d_N) \neq 0$. A similar argument yields

$$\begin{aligned}
 & \pi_N^*(\alpha'_N) - \pi_N^*(\alpha_N) - e^{d_N}(\alpha'_N - \alpha_N) \\
 &= E_{N,0} \{\phi_N(\Lambda_N, \alpha'_N) - \phi_N(\Lambda_N, \alpha_N)\} (e^{\Lambda_N} - e^{d_N}) \geq 0
 \end{aligned}$$

and hence

$$(3.3) \quad 0 \leq \pi_N^*(\alpha_N) - \pi_N(\alpha_N) \leq e^{d_N} E_{N,0} |e^{-\bar{\Delta}_N} - 1|.$$

By (2.3) there exists a sequence $\gamma_N \downarrow 0$ such that

$$E_{N,0} |e^{-\Delta_N} - 1| 1_{(\gamma_N \tau_N^{\frac{1}{2}}, A)}(|\Delta_N|) = o(\tau_N)$$

for $N \rightarrow \infty$. In view of (3.2) this implies

$$(3.4) \quad E_{N,0} |e^{-\bar{\Delta}_N} - 1| 1_{(\gamma_N \tau_N^{\frac{1}{2}}, \infty]}(|\bar{\Delta}_N|) 1_{[0, A)}(|\Delta_N|) = o(\tau_N).$$

Also (3.1) and (2.2) with $x_0 = D + 1$ yield

$$\begin{aligned}
(3.5) \quad & E_{N,0} |e^{-\bar{\Delta}_N - 1}| 1_{[0, \gamma_N \tau_N^{\frac{1}{2}}]}(|\bar{\Delta}_N|) = \\
& = O(\gamma_N \tau_N^{\frac{1}{2}} P_{N,0}(0 < |\bar{\Delta}_N| \leq \gamma_N \tau_N^{\frac{1}{2}})) \\
& = O(\tau_N^{\frac{1}{2}} P_{N,0}(0 < |\Lambda_N - d_N| \leq \tau_N^{\frac{1}{2}})) = o(\tau_N) .
\end{aligned}$$

If $\Delta_N \geq 0$ then $\bar{\Delta}_N \geq 0$ and because of (2.4)

$$(3.6) \quad E_{N,0} |e^{-\bar{\Delta}_N - 1}| 1_{[A, \infty]}(\Delta_N) \leq P_{N,0}(\Delta_N \geq A) = o(\tau_N) .$$

If $\Delta_N \leq 0$ then $\bar{\Delta}_N \leq 0$ and (2.5) ensures that

$$\begin{aligned}
(3.7) \quad & e^{d_N} E_{N,0} |e^{-\bar{\Delta}_N - 1}| 1_{[-\infty, -A]}(\Delta_N) \\
& \leq e^{d_N} E_{N,0} e^{-\bar{\Delta}_N} 1_{[-\infty, -A]}(\Delta_N) 1_{[-\infty, 0]}(\bar{\Delta}_N) \\
& \leq E_{N,0} e^{\Lambda_N} 1_{[-\infty, A]}(\Delta_N) = P_{N,1}(\Delta_N \leq -A) = o(\tau_N) .
\end{aligned}$$

Together (3.1) and (3.3) - (3.7) imply that $\pi_N^*(\alpha_N) - \pi_N(\alpha_N) = o(\tau_N)$ and the proof is complete.

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